# Stat 155 Lecture 20 Notes

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## 1 Shapley Value

### 1.1 Shapley's axioms

Here are Shapley's[1] axioms for allocation functions.

1. Efficiency: $\sum_{i=1}^{n} \psi_i(v) = v(\{1, \dots, n\})$.
2. Symmetry: If, for all $S \subseteq \{1, \dots, n\}$ and $i, j \notin S$, $v(S \cup \{i\}) = v(S \cup \{j\})$, then $\psi_i(v) = \psi_j(v)$.
3. No freeloaders: For all $i$, if for all $S \subseteq \{1, \dots, n\}$, $v(S \cup \{i\}) = v(S)$, then $\psi_i(v) = 0$.
4. Additivity: $\psi_i(v + u) = \psi_i(v) + \psi_i(u)$.

Shapley's theorem says that Shapley's axioms uniquely determine the allocation $\psi$. We call the unique allocation $\psi(v)$ the Shapley value of the players in the game defined by the characteristic function $v$.

**Theorem 1.1** (Shapley)**.** *The following allocation uniquely satisfies Shapley's axioms:*

$$\psi_i(v) = \mathbb{E}_\pi \phi_i(v, \pi),$$

*where the expectation is over uniformly chosen permutations $\pi$ on $\{1, \dots, n\}$ and*

$$\phi_i(v, \pi) = v(\pi(\{1, \dots, k\})) - v(\pi(\{1, \dots, k-1\})),$$

*where $k = \pi^{-1}(i)$.*

**Example 1.1.** For the identity permutation, $\pi(i) = i$,

$$\phi_i(v, \pi) = v(\{1, \dots, i\}) - v(\{1, \dots, i-1\}),$$

which is how much value $i$ adds to $\{1, \dots, i-1\}$. And for a random $\pi$, $\phi_i(v, \pi)$ is how much value $i$ adds to the random set $\pi(\{1, \dots, \pi^{-1}(i) - 1\})$.

[1]Lloyd Shapley was a professor of mathematics at UCLA. He won the 2012 Nobel Prize for Economics.

## 1.2 Junta games

**Example 1.2.** A Junta[2] game ($J$-veto game) is a game where there is a set $J \subseteq \{1, \ldots, n\}$ with all the power:

$$w_J(S) = \mathbb{1}_{(J \subseteq S)} = \begin{cases} 1 & J \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

For any permutation $\pi$,

$$\begin{aligned} \psi_i(w_J, \pi) &= w_J(\pi(\{1, \ldots, \pi^{-1}(i)\})) - w_J(\pi(\{1, \ldots, \pi^{-1}(i) - 1\})) \\ &= \mathbb{1}_{(i \in J, J \subseteq \pi(\{1, \ldots, \pi^{-1}(i)\}))} \\ &= \mathbb{1}_{(i \in J, \pi^{-1}(J) \subseteq \{1, \ldots, \pi^{-1}(i)\})} \\ &= \mathbb{1}_{(i \in J, \pi^{-1}(i) \in \pi^{-1}(J), \pi^{-1}(J) \subseteq \{1, \ldots, \pi^{-1}(i)\})}, \end{aligned}$$

so

$$\begin{aligned} \psi_i(w_J) &= \mathbb{E}_\pi[\phi_i(w_J, \pi)] \\ &= \mathbb{1}_{(i \in J)} \mathbb{P}(\pi^{-1}(i) = \max_{j \in J} \pi^{-1}(j)) \\ &= \mathbb{1}_{(i \in J)} \frac{1}{|J|}. \end{aligned}$$

Check that this agrees with the axioms:

1. Efficiency: $\sum_{i=1}^n \psi_i(w_J) = 1 = w_J(\{1, \ldots, n\})$.
2. Symmetry: If for all $S \subseteq \{1, \ldots, n\}$ not containing $i$ and $j$, $w_J(S \cup \{i\}) = w_J(S \cup \{j\})$ (and this is true for $i, j \in J$ and for $i, j \notin J$), then $\psi_i(w_J) = \psi_j(w_J)$.
3. Dummy: If for all $S \subseteq \{1, \ldots, n\}$, $w_J(S \cup \{i\}) = w_J(S)$ (and this is true for $i \notin J$), then $\psi_i(w_J) = 0$.

**Lemma 1.1** (Characteristic functions as Junta games)**.** *We can write any $v$ as a unique linear combination of $w_J$.*

*Proof.* Write $v$ as a vector, with one coordinate for each subset $S \subseteq \{1, \ldots, n\}$. Write a matrix $W$, with rows indexed by $S \subseteq \{1, \ldots, n\}$, columns indexed by $J \subseteq \{1, \ldots, n\}$, and entries $w_J(S)$. If we make sure these subsets are ordered by cardinality, then this matrix is lower triangular, with 1s on its diagonal. Since $W$ is invertible, we can solve the equation $v = Wc$ to obtain a unique $c$, with one entry $c_J$ for each $J \subseteq \{1, \ldots, n\}$, and then we have

$$v(S) = \sum_J w_J(S) c_J. \qquad \square$$

[2] In Spanish, a "junta" is a small group with all the power. In Latin America, this has historically occurred many times.

### 1.3 Shapley's theorem

Let's prove Shapley's theorem.

*Proof.* First, we want to show that the allocation $\psi_i(v) = \mathbb{E}_\pi[\phi_i(v, \pi)]$ satisfies Shapley?s axioms. For any $\pi$, $\psi_i(v, \pi)$ satisfies the efficiency, dummy, and additivity axioms. These axioms all involve linear expressions in $i$, so they are preserved under expectation. Symmetry follows from the randomization.

1. Efficiency:

$$\begin{aligned}
\sum_{i=1}^{n} \psi_i(v, \pi) &= \sum_{i=\pi(1)}^{\pi(n)} [v(\pi(\{1, \ldots, i\})) - v(\pi(\{1, \ldots, i-1))] \\
&= \sum_{j=1}^{n} [v(\pi(\{1, \ldots, j\})) - v(\pi(\{1, \ldots, j-1))] \\
&= v(\{1, \ldots, n\})
\end{aligned}$$

2. Dummy:

$$\pi(\{1, \ldots, \pi^{-1}(i)\}) = \pi(\{1, \ldots, \pi^{-1}(i) - 1\}) \cup \{i\},$$

so

$$\psi_i(v, \pi) = v(\pi(\{1, \ldots, i\})) - v(\pi(\{1, \ldots, i-1)) = 0.$$

To prove uniqueness, represent $v$ as a unique linear combination of Junta game characteristic functions $w_J(S) = \mathbb{1}_{(J \subseteq S)}$. Then $\psi_i(w_J) = \mathbb{1}_{(i \in J)}/|J|$ is the unique allocation satisfying the Shapley axioms for the Junta games. Additivity implies that $\psi_i(v)$ is unique. □

This proof actually gives us a nice way to compute the characteristic function. Solve for the coefficients $c_J$ in $v(S) = \sum_J c_J w_J(S)$ by solving the linear system mentioned in the Junta game lemma.

**Example 1.3.** Consider a glove game like before, with characteristic function

$$v(\{1, 2, 3\}) = v(\{1, 2\}) = v(\{1, 3\}) = 100,$$
$$v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2, 3\}) = v(\varnothing) = 0.$$

Solving the linear system, we get

$$v(S) = 100 w_{\{1,2\}}(S) + 100 w_{\{1,3\}}(S) - 100 w_{\{1,2,3\}}(S),$$

and hence

$$\psi_1(v) = 100 \left( \frac{1}{2} + \frac{1}{2} - \frac{1}{3} \right) = 100 \cdot \frac{2}{3},$$
$$\psi_2(v) = \psi_3(v) = 100 \left( \frac{1}{2} - \frac{1}{3} \right) = 100 \cdot \frac{1}{6}.$$